BH and fuzzball PT from quantum SW curves

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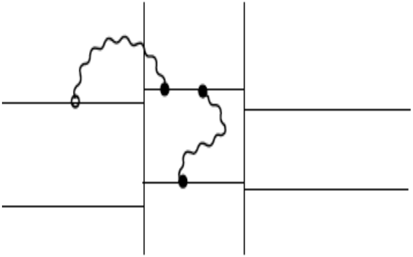
Based on

- Bianchi, Consoli, Grillo, Morales [2105.04245, 2109.09804]
- Bianchi, Di Russo [2110.09579, 2203.14900, 2207.vwxyz]
- Bianchi, Consoli, Grillo, Morales [..., ...]
- Bianchi, Grillo, Morales [..., ...]

See also

- Isomonodromic approach: Carneiro da Cunha, Cavalcante, Barragan Amado, Pallante [1702.01016; 1812.08921; 2002.06108; 1906.10638; 2109.06929]
- Exact WKB quantization, Seiberg-Witten approach: Aminov, Grassi, Hatsuda [1908.07065; 2006.06111]
- AGT correspondence: Bonelli, Iossa, Panea-Lichtig, Tanzini [2105.04483, 2201.04491]; Consoli, Fucito, Morales, Poghossian [2206.abcd]
- Integrability: Fioravanti, Gregori [1908.08030, 2112.11434]

What do they have in common?

Everything ... if M87* is a Kerr BH [in AdS]

Not much if M87* is a fuzzball [microstate]

Plan

- BHs and fuzzball perturbation theory
- Quantum Seiberg-Witten curves for $\mathcal{N} = 2$ SYM à la Nekrasov-Shatashvili
- Alday-Gaiotto-Tachikawa duality and connection formulae
- Quasi-Normal Modes and other observables
- Can we tell fuzz balls from BHs? Multipoles, echoes/QNMs, ...
- D1/D5 $\sim$ D3/D3' Circular fuzzballs: a case study
- Conclusions and outlook

Black Hole and fuzzball perturbation theory

- Choose your preferred BH or fuzzball: $D = 4, 5, 6$, flat or (A)dS, ... and your preferred perturbation: massless/massive; spin $s = 0, 1, 2$, ...
- Linearized wave-equation (e.g. $s = 0$)

$$\Box\Phi = \mu^2\Phi$$

$$\Phi = e^{-i\omega t} e^{im_\phi \phi} R(r) S(\theta) \times e^{im_\psi \psi} \times e^{ipz} \times e^{iP_i Z^i}$$

- Separate radial (r) and angular (θ) dynamics à la Carter

$$K^2 = \ell(\ell{+}D{-}3) + ...$$

- In canonical form: Schrödinger-like equation(s)

$$\Phi''(y) + Q_{BH}(y)\Phi(y) = 0$$

From Regge-Wheeler-Zerilli to Teukolsky

Radial dynamics: CHE = Confluent Heun Equation

- one irregular singularity ∞: $R \sim e^{i\omega r}$
- two regular singularities $r = r_H$, $r = 0/r_-$: $R \sim (r - r_s)^\sigma$

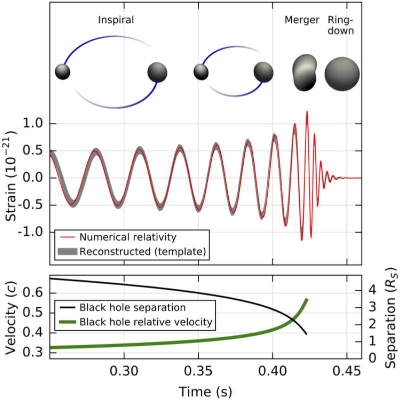
Angular dynamics

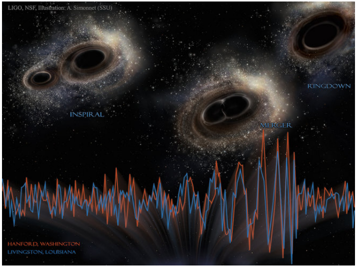
- Schwarzschild (or RN): Spherical harmonics $K^2 = \ell(\ell+D-3) = K_0^2$ independent of ω
- Kerr(-Newman): Spheroidal harmonics $K^2 = K_0^2 + \mathcal{O}(a_J\omega)$ ... CHE ... intertwined with radial

For extremal BHs $r_+ = r_-$ irregular singularity ... further confluence

Why BH/fuzzball Perturbation Theory?

Gravitational Waves from BH mergers





collaboration (GW150914)

Inspiral, Merger, Ring-down ... QNMs, ... echoes

Observables

- Quasi Normal Modes (QNMs): prompt $\sim$ photon-rings, late ... echoes

$$\omega^{WKB}_{QNM} = \omega_c(\ell, ...) - i(2n+1)\lambda$$

ω_c frequency of (un)stable circular orbits
λ Lyapunov exponent, exponentially growing geodesic deviation ...
Numerical methods: Leaver continuous fractions ... 'exact' results
Caveat: 'seed' for ω (eg WKB), extremal ... confluence

- (Near) Super-radiant (N-SR) modes $\mathrm{Im}\omega \approx 0$, amplification factors $Z_{\ell,m}(\omega)$
- Grey-body factors, absorption cross actions $\sigma_{abs}(\omega)$
- Tidal Love Numbers: static $L(0)$ vs dynamical $L(\omega)$ in $D \geq 4$

Need connection formulae ... use NS partition function and AGT correspondence

From branes to BHs and back

Hanany-Witten for 'quantum' Seiberg-Witten

- $\mathcal{N}=2$ SYM with $G=(S)U(2)$: $N_c=2$ colour D4-branes, $N_f=(2,2)$ flavour D4-branes ($\beta=0$, later on decoupling)
- 'Classical' Seiberg-Witten elliptic curve

$$qy^2 P_L(x) + yP_c(x) + P_R(x) = 0 \quad , \qquad q = \Lambda^\beta$$

with

$$P_L=(x-m_1)(x-m_2) \quad , \quad P_C=(x-a_1)(x-a_2) \quad , \quad P_R=(x-m_3)(x-m_4)$$

- Quantize à la Nekrasov-Shatasvili: $\varepsilon_1=\hbar, \varepsilon_2=0 \sim \Omega$ background

$$\hat{x} = \hbar y \partial_y \quad , \quad \hat{y} = y$$

- 'Quantum' SW curve ... 2nd order diff eq

$$\left[A(y)\hat{x}^2 + B(y)\hat{x} + C(y)\right] U(y) = 0$$

- NS prepotential

$$\mathcal{F}(a, m_f; q, \hbar) = \mathcal{F}_{tree} + \mathcal{F}_{1-loop} + \mathcal{F}_{inst}$$

- Decouple flavours $N_f \to N_f - 1$ by double scaling $q_{N_f} \to 0$, $m_{N_f} \to \infty$ with $q_{N_f-1} = q_{N_f} m_{N_f}$ fixed

Gauge/gravity encyclopedia (Radial)

Same structure as BH perturbations (Heun) ... same physics (?)
Radial equation

- D3-branes, D1-D5 (D3-D3') small BHs, ... $N_f = (0,0)$ DRDCHE
- BMPV BHs in 5-d (SUSY, extremal), ... $N_f = (0,1) \sim (1,0)$ RDCHE
- Intersecting D3's (4-charge BHs in 4-d), eKN, eSTURBHs ... $N_f = (1,1)$ DCHE
- CCLP (general 5-d charged and rotating BHs), D1-D5 (D3-D3') fuzzball (smooth, horizonless 6-d), JMaRT (smooth, horizonless 6-d), ... $N_f = (0,2) \sim (2,0)$ RCHE
- KN BHs in 4-d, STURBHs ... $N_f = (2,1) \sim (1,2)$ CHE
- KN BHs in AdS$_4$ with $\mu_\Phi^2 L^2 = -2$ ($\Delta = 1, 2$) $N_f = (2,2)$ HE

H=Heun, E=Equation, C=confluent, R=reduced, D=doubly
All with $N_c = 2$ and $N_L, N_R \leq 2$!
Some enjoy generalized Couch-Torrence (CT) conformal inversions

Gauge/gravity encyclopedia (Angular) and dictionary

Angular equation $\longrightarrow$ 'spheroidal' harmonics

- All 4-d geometries (S^2): $N_f = (1,2)$ CHE
- All 5-d and 5-d $\times S^1$ geometries (S^3): $N_f = (0,2)$ RCHE

More precisely

- RG-scale / instanton counting parameter $q = \Lambda^\beta \sim (\omega a_J)^\beta$, $\beta = 4 - N_f$, $a_J \sim J/M$ BH/fuzz spin
- Coulomb-branch variable $u = \langle Tr\phi^2 \rangle = a^2 + ... \sim K^2$ (Carter) separation constant
- Masses of Hypermultiplets $m_f \sim m_\phi, m_\psi$ angular momenta
- QNM quantization condition: $a = \hbar(\ell + \frac{1}{2})$
- 'Straightforward' determination of K^2

$$\frac{1}{4}(1+K^2) = u = -q\partial_q \mathcal{F}_{NS}(a = \ell + \frac{1}{2}, m_f, q; \hbar) = a^2 + ...$$

  using 'quantum' Matone relation

Radial dictionary

- RG-scale / instanton counting parameter $q = \Lambda^\beta \sim (\omega M)^\beta$, $\beta = 4 - N_f$, M BH/fuzz mass-scale
- Coulomb-branch variable $u = \langle Tr\phi^2 \rangle = a^2 + ... \sim K^2 + \delta K^2$ shifted (Carter) separation constant
- Masses of Hypermultiplets $m_f \sim m_\phi + \delta m_\phi, m_\psi + \delta m_\psi$ shifted angular momenta (impact parameters)
- QNM quantization condition: n 'overtone number'

$$a_D = -\frac{1}{2\pi i} \partial_{a_R} \mathcal{F}_{NS}(a, m_f, q; \hbar) = n\hbar$$

NOT straightforward: a_γ ... photon-rings ... critical geodesics .. WKB

- Invert quantum Matone relation or use 'difference' equation ($\hat{x} \leftrightarrow \hat{y}$)

$$u = -q\partial_q \mathcal{F}_{NS}(a, m_f, q; \hbar) = a^2 + ...$$

to get $a(u)$ with $u_R = u_A + \delta u_{AR}$, plug into expression for a_D

- Solve for ω in terms of n, ℓ, m's, M, Q, J's ... compare with WKB, Leaver

The AGT picture

AGT duality [Alday, Gaiotto, Tachikawa]: 4-d $\mathcal{N}=2$ quiver theories $\sim$ 2-d Liouville CFT with $c = 1 + 6Q^2$, $Q = b + \frac{1}{b}$, $b = \sqrt{\varepsilon_1/\varepsilon_2}$
Conformal blocks $\sim$ (ratio of) NS quiver partition functions

$$\mathcal{C}^{\alpha_1\cdots\alpha_{n+1}}_{p_0\cdots p_{n+1}}\left(\{z_i\}\right)\prod_{j=1}^{n} z_j^{-\Delta_{p_j}+\Delta_j+\Delta_{p_j+1}} = \frac{Z_{\text{inst}}\left(\{\vec{a}_i\},\{q_i\}\right)}{Z_{U(1)}(\{q_i\})}$$

Consider $SU(2)\times SU(2)$ quiver ($n=2$), 'wave-function'

$$\Psi(\{z_i\}) = \mathcal{C}^{\alpha_1\cdots\alpha_3}_{p_0\cdots p_3}\left(\{z_i\}\right)$$

For $\alpha_3 = -b/2$ $\left(L_{-1}^2 + b^2 L_2\right) V_{\alpha_3} \sim 0$ (null): BPZ equation

$$\Psi''(\{z_i\}) + b^2 \sum_{i\neq 1}^{4}\left[\frac{\Delta_i}{(z-z_i)^2} + \frac{1}{z-z_i}\partial_{z_i}\right]\Psi(\{z_i\}) = 0$$

set $z_0 = \infty$, $z_1 = 1$, $z_2 = q$, $z_3 = y$, $z_4 = 0$,
double scaling limit $b \to 0$ ($\varepsilon_1 = 0$ NS Ω-background) with fixed

$$b^2\Delta_i = \delta_i\,, \quad b^2 c_i = \nu_i\,, \quad \partial_{z_i}\Psi(y;\{z_i\}) = c_i\Psi(y;\{z_i\})$$

get quantum SW curve for $\Psi(y)$ with $N_f = 4 \sim$ KN BH in AdS

Connection formulae from fusing and braiding

Solution near $z_1 = 1$ ('horizon' or 'cap')

$$\Psi^{(1)}(z) = \sum_{\alpha} C^{(1)}_{\alpha} F^{(1)}_{\alpha}(1-z)$$

- BHs: $C^{(1)}_{+} = 0$ no outgoing signal
- fuzzballs/ECO: $C^{(1)}_{-} = 0$ regularity / no ingoing signal

Solution near $z_2 = 0$ ('infinity')

$$\Psi^{(2)}(z) = \sum_{\alpha} C^{(2)}_{\alpha} F^{(2)}_{\alpha}(z)$$

For QNM $C^{(2)}_{-} = 0$: only outgoing
Connection formulae for D-R-D-C-HE (... so far largely unknown)

$$F^{(2)}_{\alpha}(z) = \sum_{\beta} M_{\alpha\beta} F^{(1)}_{\beta}(1-z)$$

using fusing and braiding matrices [Bonelli, Iossa, Panea-Lichtig, Tanzini; Consoli, Fucito, Morales, Poghossian; ....]

E.g. connection formulae for RCHE $\sim N_f = (2,0)$

For instance (e.g. circular fuzzball, ...), RCHE $\sim N_f = (2,0)$ qSW ($\beta_{SYM} = 2$)

$$\frac{d^2W}{dz^2} + \left(\frac{\gamma}{z} + \frac{\delta}{z-1}\right)\frac{dW}{dz} + \frac{\beta z - \zeta}{z(z-1)}W = 0$$

gauge / Heun dictionary

$$u = \zeta - \beta + \frac{1}{4}(\gamma + \delta - 1) = a^2 + \dots, \quad \beta = q = \Lambda^2\,, \quad \gamma = \mu_\psi \sim a_0\,, \quad \delta = \mu_\phi \sim a_1$$

From $z = 0$ (regular) to $z = \infty$ (irregular) (e.g. $z = -\rho^2/a_f^2$)

$$\sqrt{\pi} z^{+\frac{1}{4}+\frac{\gamma+\delta}{2}} e^{\pm i\delta\frac{\pi}{2}} W_0(\zeta,\beta,\gamma,\delta;z) =$$

$$\sum_{\sigma=\pm}\frac{a(\zeta)\Gamma(2\sigma a(\zeta))^2\Gamma(\gamma)(e^{i\pi}\beta)^{-\frac{1}{4}-\sigma a(\zeta)}e^{-\frac{1}{2}\partial_{a_0}\mathcal{F}+\frac{\sigma}{2}\partial_a\mathcal{F}}}{\sigma\Gamma\left(\frac{\gamma-\delta+1}{2}+\sigma a(\zeta)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}+\sigma a(\zeta)\right)}e^{2i\sqrt{\beta z}}W_\infty(\zeta,\beta,\gamma,\delta;\frac{1}{\sqrt{z}})$$

$+$

$$\sum_{\sigma=\pm}\frac{a(\zeta)\Gamma(2\sigma a(\zeta))^2\Gamma(\gamma)(e^{-i\pi}\beta)^{-\frac{1}{4}-\sigma a(\zeta)}e^{\frac{\sigma}{2}\partial_a\mathcal{F}-\frac{1}{2}\partial_{a_0}\mathcal{F}}}{\sigma\Gamma\left(\frac{\gamma-\delta+1}{2}+\sigma a(\zeta)\right)\Gamma\left(\frac{\gamma+\delta-1}{2}+\sigma a(\zeta)\right)}e^{-2i\sqrt{\beta z}}W_\infty(\zeta,e^{2\pi i}\beta,\gamma,\delta;\frac{1}{\sqrt{z}})$$

where $\mathcal{F}(a,q,m_f;\hbar)$ NS pre-potential

Obsevables from ‘new’ connection formulae

Other observables from conserved current/flux

For real ω $Q = Q^*$: conserved 'current' [Bonelli, Iossa, Panea-Lichtig, Tanzini; Consoli, Fucito, Morales, Poghossian]

$$\mathcal{J} = \mathrm{Im}[\Psi^* \partial_z \Psi] = k_H(|C_+^{(1)}|^2 - |C_-^{(1)}|^2) = k_\infty(|C_+^{(2)}|^2 - |C_-^{(2)}|^2)$$

$$\mathcal{J}_{abs} = -k_H|C_-^{(1)}|^2 \quad , \quad \mathcal{J}_{ref} = +k_H|C_+^{(1)}|^2$$

$$\mathcal{J}_{in} = -k_\infty|C_-^{(2)}|^2 \quad , \quad \mathcal{J}_{out} = +k_\infty|C_+^{(2)}|^2$$

- Absorption cross section / grey body factor

$$\sigma_{abs}(\omega) = \frac{|\mathcal{J}_{abs}(\omega)|}{|\mathcal{J}_{in}(\omega)|} = \frac{k_H|C_-^{(1)}|^2}{k_\infty|C_-^{(2)}|^2} \sim \exp(-a_D/\hbar)$$

- Echoes: present when reflectivity $\mathcal{R} \neq 0$ and $\mathcal{E} = \mathcal{R}/(C_+^H - \mathcal{R}C_-^H) \neq 0$

  $$G(z,z') = \theta(z-z')\psi_{in}(z)\psi_{out}(z') + \theta(z'-z)\psi_{in}(z')\psi_{out}(z) + \mathcal{E}\psi_{out}(z)\psi_{out}(z')$$

  with solutions of homogeneous eq: $\psi_{in}|_H = F_-$ and $\psi_{out}|_\infty = \tilde{F}_+$
- Amplification factor (super-radiance)

$$Z_{\ell,m,s}(\omega) = \frac{|\mathcal{J}_{out}(\omega)|}{|\mathcal{J}_{in}(\omega)|} - 1 = \frac{k_H|C_-^{(1)}|^2}{k_\infty|C_-^{(2)}|^2}$$

Super-radiant modes

Super-radiant threshold

$$\omega_{SR} \leq m_\phi \Omega_H$$

For KN BHs $J = Ma_J$, $M^2 \geq a^2 + Q^2$

$$\Omega_H^{KN} = \frac{a_J}{a_J^2 + r_+^2}$$

For STURBHs $M{=}\frac{1}{4}\sum_i \sqrt{m^2{+}Q_i^2}$, $Q_i{=}2ms_ic_i$, $m{\geq}a$, $s_i{=}\sinh\xi_i$, $c_i{=}\cosh\xi_i$

$$\Omega_H^{STURBH} = \frac{a}{2m[r_+\Pi_c + (2m - r_+)\Pi_c]}$$

where $\Pi_s = \Pi_i s_i$, $\Pi_c = \Pi_i c_i$
At extremality $r_+ = r_- = r_H$ ... decoupling limit $N_f = (2,1) \to N_f = (1,1)$

$$q \to 0\,, \quad m_2 \to \infty\,, \quad q\, m_2 = \tilde{q}$$

super-radiant modes $\mathrm{Im}\omega \approx 0$ zero-damping modes (ZDMs)

Near extremality

$$\omega = \omega_{SR} + \nu\delta_r \quad , \quad \delta_r = r_+ - r_- << r_\pm$$

Impose $\Psi \sim e^{i\omega_{SR} r}$ (outgoing at infinity) and $\Psi \sim (r - r_H)^{\frac{1}{2}+\alpha}$ (regular at horizon) and match using only tree-level and 1-loop terms in $\mathcal{F}_{NS}$ ... $a_D = \hbar\, n$

$$\exp\left[-\frac{2\pi i a_D}{\hbar}\right] = 1 = \left(-\frac{q}{\hbar}\right)^{\frac{2\sqrt{u}}{\hbar}} \frac{\Gamma\left(1+\frac{\sqrt{u}}{\hbar}\right)^2}{\Gamma\left(1-\frac{\sqrt{u}}{\hbar}\right)^2} \prod_{i=1}^{3} \frac{\Gamma\left(\frac{1}{2}+\frac{m_i-\sqrt{u}}{\hbar}\right)}{\Gamma\left(\frac{1}{2}+\frac{m_i+\sqrt{u}}{\hbar}\right)}$$

get [MB, Consoli, Grillo, Morales]

$$\omega_{\rm NSR}^{neKN} = \omega_{\rm SR}\left(1 + 4\pi r_+ T_{\rm BH}\right) - 2\pi i\, T_{\rm BH}\left(\alpha + n + \frac{1}{2}\right)$$

with $T_{\rm BH} = \frac{\delta_r}{4\pi(r_+^2 + a_{\mathcal{J}}^2)}$ and $\alpha^2 = A + \frac{1}{4} - (a_{\mathcal{J}}^2 + 6r_+^2)\omega_{\rm SR}^2$

For neSTURBHs [MB, Di Russo]

$$\omega_{\rm NSR}^{neSTURBH} = \omega_{\rm SR} + \Omega_H\delta\left[\omega_{\rm SR}(\Pi_c - \Pi_s) - i(n + \frac{1}{2} + \alpha)\right] + \ldots$$

with $\alpha^2 = A + \frac{1}{4} - a_{\mathcal{J}}^2\omega_{\rm SR}^2[7 + 6\sum_i \sigma_i^2 + 4\sum_{i<j}\sigma_i^2\sigma_j^2]$

Intermezzo: Couch-Torrence conformal inversions

Extremal Reissner-Nordström BHs ($Q = M$), with $u = r - Q$ symmetry under conformal inversions [Couch, Torrence]

$$u \to Q^2/u \quad , \quad ds^2_{extr} \to W(u) ds^2_{extr}$$

exchange horizon $u_H = 0$ ($r_H = Q$) with infinity,
Photon-sphere $u_c = Q$ ($r_c = 2Q$) fixed! [MB, Di Russo]
Deflection angle: scattering = fall! Similar to "B2B" [Kälin, Porto; Di Vecchia, Russo, Veneziano; ...]
BUT no analytic continuation needed [MB, Di Russo]

$$\Delta\phi_{fall}(J, E) = \int_0^{u_i} \frac{J du}{u^2 P_u(u; J, E)} = \int_{Q^2/u_i}^{\infty} \frac{J du}{u^2 P_u(u; J, E)} = \Delta\phi_{scatt}(J, E)$$

Valid for other extremal geometries in $D \geq 4$ and for massive (BPS) probes

- D3-branes $u_c = L$
- D3-D3' (D1-D5) 'small' BHs $u_c^2 = L_3 L_{3'}$
- 4-charge STU BHs with $Q_1 Q_2 = Q_3 Q_4$ or permutations, $u_c^4 = Q_1 Q_2 Q_3 Q_4$

Rotating BHs and branes and photon-halos

For extremal KN, generalised Couch-Torrence conformal inversions: symmetry of radial equation, NOT of metric [Couch, Torrence]
Fixed locus depends on impact parameters ($b = K/E$, $b_z = J_z/E$)
Critical 'radius' not fixed: $r_c \in [r_{min}, r_{Max}]$ ... photon-halo/light-shell
For eSTURBHs (nonBPS): generalised CT [Cvetic, Pope, Saha], when [MB, Di Russo 2203.14900]

$$(Q_1 + \sqrt{Q_1^2 + a^2})(Q_2 + \sqrt{Q_2^2 + a^2}) = (Q_3 + \sqrt{Q_3^2 + a^2})(Q_4 + \sqrt{Q_4^2 + a^2})$$

Equality of radial actions, same E, J, J_z (again no analytic continuation) Yet $\Delta\phi_{fall} \neq \Delta\phi_{scatt}$ due to (divergent) bdry terms in $\partial S/\partial J_z$ at $u_H = 0$

Can we tell a fuzzball from a (putative) BH ?

Fuzzball spectroscopy: Ringdown, QNMs, and echoes

Multi-center micro-states in $D = 4(5)$ much more involved

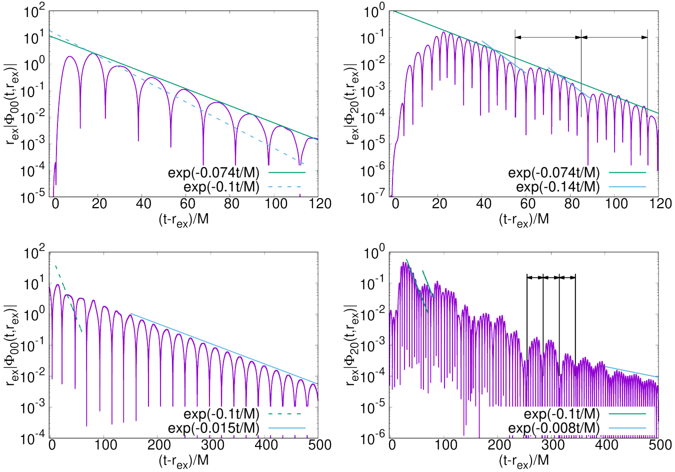
- Generic case, only one isometry $E \sim \partial_t$ ... numerical methods
- Axi-symmetric case, $J \sim \partial_\phi$ conserved, yet not separable (r, θ) vs (ρ, z) ... some progress
- Axi-symmetric case with 'equatorial' $z \leftrightarrow -z$ symmetry ... $m = \ell$ tractable ...

Time evolution of a massless scalar Gaussian shell

$$\Phi(t = 0, \vec{x}) = A e^{\frac{(r-r_0)^2}{\sigma^2}}$$

Intricate mode mixing, issue with 4-d singularities ... numerical analysis: evidence for echoes ! [MB, Consoli, Grillo, Ikeda, Morales, Pani, Raposo]

Time evolution of Scalar field in 3-center micro state geometry



Modes: $\ell = 0, m = 0$ (Left) and $\ell = 2, m = 0$ (Right); $L = 0.67M$, $\kappa = 1$ (Top), and $L = 0.27M$, $\kappa = 2$ (Bottom). Gaussian shell with $\sigma = 0.67M$ (Top) and $\sigma = 0.27M$

Multipolar structure: BHs vs fuzzballs/ECOs

In GR 'no-hair theorem': Kerr(-Newman) BH $J = Ma$,
very peculiar multipolar structure [Geroch, Hanssen, Thorne]

$$\mathcal{M}_\ell + i\mathcal{S}_\ell = M(ia)^\ell \quad : \quad \mathcal{M}_{2\ell+1} = 0 \ , \quad \mathcal{S}_{2\ell} = 0$$

ECOs/fuzzballs not excluded by GW observations: especially GW190814 (too light lighest body) and GW190521 (at least one of the two bodies too massive)
Generic micro-state geometries: no axial, no equatorial symmetry ... much richer and involved multipolar structure ... see plots

$$\mathcal{M}_{2\ell+1} \neq 0 \quad , \quad \mathcal{S}_{2\ell} \neq 0$$

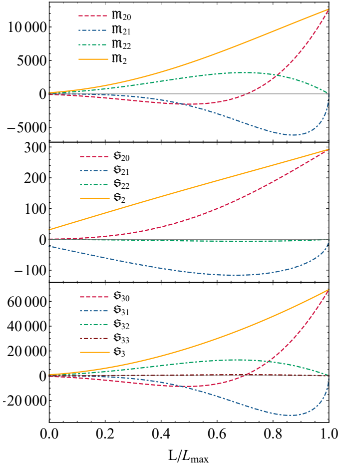
Breaking of equatorial symmetry [Bena, Mayerson; MB, Consoli, Grillo, Morales, Pani, Raposo; ...]
The big issues

- Only a (small) fraction of micro-states known
- Averaging procedure ... BH results (?)

... measure ... toy model (2-charge small BHs and circular fuzz balls)

E.g. 3-center microstates in STU SUGRA

Comparison for $\kappa = (325, 751, 798, 272)$ $L_{Max} = 79.3361$



Invariants $\widehat{\mathcal{M}}_2$, $\widehat{\mathcal{M}}_{2,m}$ (top), $\widehat{\mathcal{S}}_2$, $\widehat{\mathcal{S}}_{2,m}$ (middle), $\widehat{\mathcal{S}}_3$, $\widehat{\mathcal{S}}_{3,m}$ (bottom) vs $L/L_{Max} \leq 1$

For fuzz balls: larger than Kerr for $L \sim L_{Max}$, smaller for $L << L_{Max}$.

$L \rightarrow 0$ limit small but non-vanishing

A case study: QNMs of 2-charge micro-states

Toy model: a_f 'radius', $M = Q_1 + Q_5$, $Q_1 = L_1^2$, $Q_5 = L_5^2$

$$ds^2 = H^{-1}\Big[-(dt+\omega_\phi d\phi)^2+(dz+\omega_\psi d\psi)^2\Big]+H\Big[(\rho^2+a_f^2\cos^2\theta)\left(\frac{d\rho^2}{\rho^2+a_f^2}+d\theta^2\right)$$

$$+\rho^2\cos^2\theta d\psi^2+(\rho^2+a_f^2)\sin^2\theta d\phi^2\Big]$$

where $H = \sqrt{H_1 H_5}$ and

$$H_i = 1+\frac{L_i^2}{\rho^2+a_f^2\cos^2\theta} \quad , \quad \omega_\phi = \frac{a_f L_1 L_5 \sin^2\theta}{\rho^2+a_f^2\cos^2\theta} \quad , \quad \omega_\psi = \frac{a_f L_1 L_5 \cos^2\theta}{\rho^2+a_f^2\cos^2\theta}$$

Scalar wave equation (both $\mu = 0$ and, if $L_1 = L_5$, $\mu \neq 0$) separates [Lunin, Mathur; MB, Consoli, Morales]:

$$\left\{\frac{1}{\rho}\partial_\rho\Big[\rho(\rho^2+a_f^2)\partial_\rho\Big]+\tilde\omega^2(\rho^2+a_f^2)\left(1+\frac{L_1^2+L_5^2}{\rho^2+a_f^2}\right)+\frac{\mathcal{L}_\phi^2}{\rho^2+a_f^2}-\frac{\mathcal{L}_\psi^2}{\rho^2}-K^2\right\}R(\rho)=0$$

$$\left[\frac{1}{\sin 2\theta}\partial_\theta\Big(\sin 2\theta\partial_\theta\Big)-\frac{m_\phi^2}{\sin^2\theta}-\frac{m_\psi^2}{\cos^2\theta}-\tilde\omega^2 a_f^2\sin^2\theta+K^2\right]S(\theta)=0$$

where $\tilde\omega^2 = E^2 - P^2$, $\mathcal{L}_\psi = a_f J_\psi - P L_1 L_5$, $\mathcal{L}_\psi = a_f J_\phi - E L_1 L_5$

Scalar wave equation, separation and qSW curves

Setting $\frac{L_1^2+L_5^2}{a_f^2} = \lambda \quad , \quad \lambda_{\phi,\psi} = \frac{\mathcal{L}^2_{\phi,\psi}}{a_f^2} \quad , \quad \lambda_\pm = \lambda_\phi \pm \lambda_\psi$

Radial equation in canonical form $\frac{\rho^2}{a_f^2} = y_R (= -z_R)$

$$\psi_R'' + \left[\frac{1-\lambda_\psi}{4y^2} + \frac{1-\lambda_\phi}{4(1+y)^2} + \frac{2+K^2-\lambda_+-a_f^2\tilde{\omega}^2\lambda}{4(1+y)} + \frac{\lambda_++a_f^2\tilde{\omega}^2(1+\lambda)-2-K^2}{4y}\right]\psi_R = 0$$

Gauge/gravity dictionary: $N_f = (2,0)$ qSW

$$q_R = \frac{\hbar^2}{4}\tilde{\omega}^2 a_f^2 \quad , \quad u_R = \frac{\hbar^2}{4}[1+K^2 - \tilde{\omega}^2(L_1^2+L_5^2)] \quad , \quad m^R_{1,2} = \frac{\hbar}{2}\frac{\mathcal{L}_\phi \mp \mathcal{L}_\psi}{a_f}$$

Angular equation in canonical form $y_A = -\cos^2\theta = -z_A$

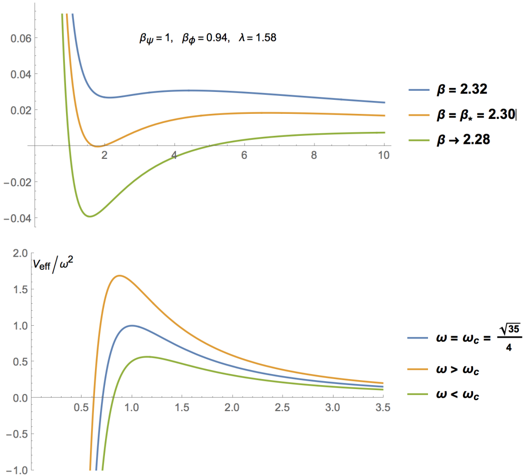
$$\psi_A'' + \left[\frac{2+K^2-m_\phi^2 - m_\psi^2}{4(1+y)} + \frac{1-m_\psi^2}{4y^2} + \frac{1-m_\phi^2}{4(1+y)^2} + \frac{a_f^2\tilde{\omega}^2-2-K^2+m_\phi^2+m_\psi^2}{4y}\right]\psi_A = 0$$

Gauge/gravity dictionary: $N_f = (2,0)$ qSW

$$q_A = \frac{\hbar^2}{4}\tilde{\omega}^2 a_f^2 = q_R! \ , \quad u_A = \frac{\hbar^2}{4}(1+K^2) = u_R + \frac{\hbar^2}{4}\tilde{\omega}^2(L_1^2+L_5^2) \ , \quad m^A_{1,2} = \frac{\hbar}{2}(m_\phi \pm m_\psi)$$

Radial = Angular for $L_1 = L_5 = M = 0$ (flat space-time!). For $a_f = 0$ small BHs

Effective potentials: fuzzball vs 'small' BH



D1/D5 $\sim$ D3/D3' QNMs $\mu = 0$

Some results in the $a_f \to 0$ BH limit

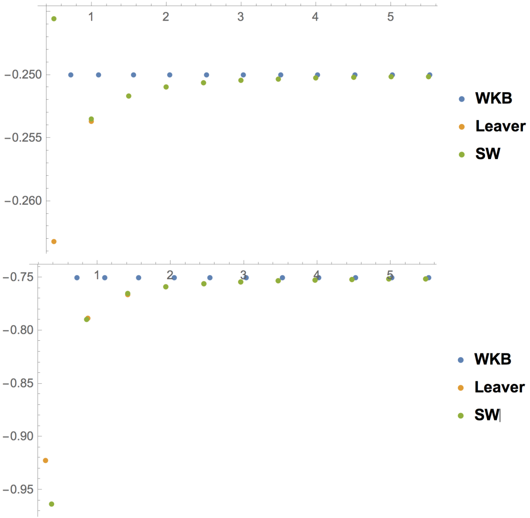
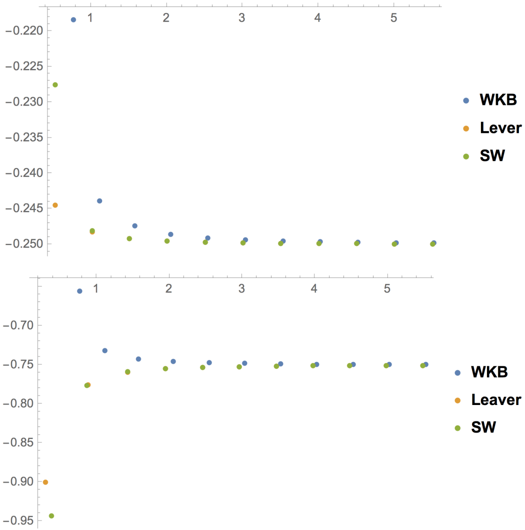


Figure: QNM of D1/D5 branes ($L = 1$) with massless scalar for $n = 0$ (top) and $n = 1$

D1/D5 $\sim$ D3/D3' QNMs $\mu \neq 0$



QNM of D1/D5 small BH ($L = 1$) for massive probe ($\mu^2 = 0.3$) with $n = 0$ (top) and $n = 1$ (bottom) for ℓ varying from 0 to 10.

Conclusions and Outlook

- BH perturbations and qSW curves are one and the same thing ... fuzzballs/ECOs are another story.
- Mathematics? Physics? Duality between M5's and M2's: wrapping M5 on $\mathbf{R}^4_\Omega \times \Sigma$ AGT correspondence ... wrapping M2 on Σ (BPS) BH ...
- Can we discriminate fuzz balls from BHs? Multipolar structure, Ring-down ... echoes, tidal Love numbers ...
- How to construct and superpose micro states?
- 2-charge D1/D5 $\sim$ D3/D3' toy model: dual to (p, w), qSW integrable, connection formulae ... different branches of QNMs and echoes
- 3-charge superstrata [Bena, Giusto, Russo, Shigemori, Warner] NOT integrable in general
- Neutral micro-states (with dipole moments) [Bah Heidmann Weck 2203.12625] ... neither
- Breaking of equatorial symmetry [Bena, Mayerson, ..], keeping axial symmetry may be the next step ...